Name: ___________________________ ID Number: ___________________________
Section Number: __________________ Serial Number: _________________________
Class Time: ______________________ Instructor’s Name: ______________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 6 pages of problems (Total of 8 Problems)

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1. (8 points) Find the value of \( a \) that makes the following function

\[
f(x) = \begin{cases} 
  a - x & \text{if } x \leq -1 \\
  x + 1 & \text{if } x > 1
\end{cases}
\]

continuous everywhere.

- \( f(x) = a - x \) is continuous since it is a polynomial
- \( f(x) = x + 1 \) is continuous since it is a polynomial
- We will be continuous at \( x = -1 \):\n
\[
\lim_{{x \to -1^-}} f(x) = \lim_{{x \to -1^+}} (a - x) = a - 1
\]

\[
\lim_{{x \to -1^-}} f(x) = \lim_{{x \to -1^+}} (x + 1) = -1 + 1 = 0
\]

\[
\Rightarrow a - 1 = 0 \Rightarrow a = 1
\]

2. Find each limit or show that it does not exist.

(a) (5 points) \( \lim_{{x \to 0^+}} \left( \frac{1}{x} - \frac{1}{|x|} \right) \).

\[
= \lim_{{x \to 0^+}} \left( \frac{1}{x} - \frac{1}{x} \right) = 0
\]

(b) (6 points) \( \lim_{{t \to 1}} \frac{\sin(t^2 - 1)}{3t^2 - 3} \).

\[
= \lim_{{t \to 1}} \frac{\sin(t^2 - 1)}{t^2 - 1} \cdot \frac{t^2 - 1}{3(t^2 - 1)} \]

\[
= \lim_{{t \to 1}} \frac{\sin(t^2 - 1)}{t^2 - 1} \cdot \frac{(t-1)(t+1)}{3(t-1)(t+1)} \]

\[
= \lim_{{t \to 1}} \frac{\sin(t^2 - 1)}{t^2 - 1} \cdot \frac{t+1}{3(t+1)} \]

\[
= \frac{1}{3} \cdot \left( \frac{3}{2} \right) = \frac{1}{2}
\]
c) (8 points) \[ \lim_{x \to -2} \frac{2x + \sqrt{4x^2 + 3x - 2}}{2x - \sqrt{4x^2 + 3x - 2}} = \lim_{x \to -2} \frac{2x + \sqrt{4x^2 + 3x - 2}}{2x - \sqrt{4x^2 + 3x - 2}} \]

(\ref{eq:1})

\[ = \lim_{x \to -2} \frac{-3x + 2}{2x - 2\sqrt{4x + 3} - \frac{2}{x} - 2} \]

(\ref{eq:2})

\[ = \lim_{x \to -2} \frac{-3x + 2}{2 + \sqrt{4 + \frac{3}{x} - \frac{2}{x^2}}} \]

(\ref{eq:3})

\[ = \lim_{x \to -2} \frac{-3x + 2}{2 + \sqrt{4 + \frac{3}{x} - \frac{2}{x^2}}} \]

(\ref{eq:4})

\[ = \frac{-3 + \frac{2}{x}}{2 + \sqrt{4 + \frac{3}{x} - \frac{2}{x^2}}} \]

(\ref{eq:5})

\[ = \frac{-3 + 0}{2 + \sqrt{4 + 0} - 0} = \frac{-3}{4} \]

(\ref{eq:6})

3. (10 points) Find a number \( k \) so that the following limit exists. Then find the limit for the found value of \( k \).

Note that Den. \( \to 0 \) as \( x \to -2 \). If Num \( \to 0 \) when \( x \to -2 \), then the given limit equals \( \pm \infty \) (i.e., DNE). So for the limit to exist, we must have

\[ \lim_{x \to -2} 3x^2 + kx + k + 3 = 0 \]

(\ref{eq:3})

\[ \Rightarrow \quad 12 - 2k + k + 3 = 0 \]

(\ref{eq:2})

\[ \Rightarrow \quad k = 15 \]

(\ref{eq:2})

When \( k = 15 \), we have

\[ \lim_{x \to -2} \frac{3x^2 + 15x + 18}{x^2 + 3x - 2} = \lim_{x \to -2} \frac{3(x+2)(x+3)}{(x+2)(x-1)} \]

(\ref{eq:1})

\[ = \lim_{x \to -2} \frac{3(x+3)}{(x-1)} \]

(\ref{eq:2})

\[ = \frac{3(4)}{-3} = -4 \]

(\ref{eq:2})
4. Let \( f(x) = \frac{\ln(2x+1)}{x^2 - 4} \)

a) (8 points) Where is \( f \) continuous?

It is continuous in its domain:

1. \( ax+1 > 0 \Rightarrow x > -\frac{1}{2} \)  ②
2. \( x^2 - 4 \neq 0 \Rightarrow x \neq \pm 2 \)  ②

So \( f \) is continuous on \((-\frac{1}{2}, 2) \cup (2, \infty)\).  ④

b) (10 points) Find all vertical asymptotes of \( f \).

Justify your answer using limits.

- We check \( x = -\frac{1}{2} \) (as it is a V.A. for \( \ln(2x+1) \))
  \[
  \lim_{x \to -\frac{1}{2}^+} \frac{\ln(2x+1)}{x^2 - 4} = \frac{-\infty}{-\frac{1}{4} - 4} = \infty \quad (4)
  \]

- We check the zeros of the denominator: \( x = \pm 2 \). But \( x = -2 \) is not on the boundary of the domain (see part (a)), then we check only \( x = 2 \).
  \[
  \lim_{x \to 2^+} \frac{\ln(2x+1)}{x^2 - 4} = \infty \quad (4)
  \]

So the V.A. are
\[ x = -\frac{1}{2} \text{ and } x = 2. \quad (1)+(1) \]
5. Consider the curve \( y = \frac{1}{x+1}, \ x < 0, \ x \neq -1. \)

a) (8 points) Use limits to find the slope of the curve at any point \( x = a \neq -1. \)

\[
\text{slope} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
= \lim_{h \to 0} \frac{\frac{1}{a+h+1} - \frac{1}{a+1}}{h}
= \lim_{h \to 0} \frac{1}{h} \cdot \frac{-h}{(a+h+1)(a+1)}
= \lim_{h \to 0} \frac{-1}{(a+h+1)(a+1)}
= \frac{-1}{(a+1)^2}, \quad a \neq -1
\]

b) (6 points) Find all points at which the slope equals \(-\frac{1}{4}\).

\[
\text{slope} = -\frac{1}{4}
\]

\[
\Rightarrow \frac{-1}{(a+1)^2} = -\frac{1}{4}, \quad \text{from part (a)}
\]

\[
\Rightarrow (a+1)^2 = 4
\]

\[
\Rightarrow a+1 = \pm 2
\]

\[
\Rightarrow a = -3 \quad \text{or} \quad a = 1
\]

\[
\Rightarrow a = -3 \quad \text{since} \ x < 0
\]

The required point is \((-3, \ -\frac{1}{2})\).
6. (10 points) Use the graph of \( f(x) = x^2 + 1 \) to find a number \( \delta > 0 \) such that for all \( x, \)

\[
0 < |x - 1| < \delta \Rightarrow |f(x) - 2| < \frac{1}{4}
\]

[Note that \( \sqrt{2} \approx 1.41, \sqrt{3} \approx 1.73, \sqrt{5} \approx 2.24, \sqrt{7} \approx 2.65 \)]

\[
\lim_{x \to 1} f(x) = 2, \quad \varepsilon = \frac{1}{4}
\]

From the graph

\[
\begin{align*}
\text{if } x_1 = \frac{3}{4} & \Rightarrow x_1 + 1 = \frac{7}{4} \Rightarrow \frac{x_1}{3} = \frac{7}{4} \Rightarrow x_1 = \frac{\sqrt{3}}{2} \quad (1) \\
\text{if } x_2 = \frac{5}{4} & \Rightarrow x_2 + 1 = \frac{9}{4} \Rightarrow x_2 = \frac{5}{2} \quad (2)
\end{align*}
\]

\[
S_1 = 1 - x_1 = 1 - \frac{\sqrt{3}}{2} \approx 0.865 = 0.135 (1)
\]

\[
S_2 = x_2 - 1 = \frac{5}{2} - 1 \approx 1.2 - 1 = 0.12 (2)
\]

We may choose

\[
S = \min (S_1, S_2) \quad (1)
\]

\[
= \min (0.135, 0.12) = 0.12 \quad (1)
\]

(or any smaller positive number)

7. (8 points) Use the Intermediate Value Theorem to show that the equation

\[
x^2 = 3x - 1
\]

has a positive solution.

Let \( f(x) = x^2 - 3x + 1 \)

Choose the interval \([1, 3]\)

Since \( f \) is continuous on \([1, 3]\)

\[
\begin{align*}
\text{if } x = 1 & \Rightarrow f(1) = 1^2 - 3(1) + 1 = -1 < 0 \\
\text{if } x = 3 & \Rightarrow f(3) = 3^2 - 3(3) + 1 = 19 > 0
\end{align*}
\]

Then, by IVT, there is a number \( c \in (1, 3) \) such that

\[
f(c) = 0
\]

\[
c = \frac{3c - 1}{c} > 0
\]

So the given equation has a positive solution.
8. \textbf{(13 points)} Sketch the graph of a function \( f \) that satisfies the following conditions:

(i) \( \lim_{x \to -\infty} f(x) = 0 \)

(ii) \( \lim_{x \to 0^+} f(x) = 1 \)

(iii) \( \lim_{x \to 0^-} f(x) = \infty \)

(iv) \( \lim_{x \to 2^-} f(x) = -\infty \)

(v) \( f \) has a removable discontinuity at \( x = 4 \).

(vi) \( f(4) = 1 \)

(vii) \( \lim_{x \to \infty} f(x) = \infty \)