An introduction of the HDG methods. Part III

Bernardo Cockburn

University of Minnesota, KFUPM

KFUPM
Dec. 25, 2012
1. The heat equation
2. The wave equation
3. Convection-diffusion
4. Linear elasticity
5. The Stokes flow
6. Incompressible fluid flow
7. Compressible fluid flow
8. Gas dynamics
9. Ongoing work and open problems
10. References
The model problem.

Consider the model problem:

\[ \begin{align*}
 c \mathbf{q} + \nabla u &= 0 \quad \text{in } \Omega \times (0, T), \\
 u_t + \nabla \cdot \mathbf{q} &= f \quad \text{in } \Omega \times (0, T), \\
 \hat{u} &= u_D \quad \text{on } \partial \Omega \times (0, T), \\
 u &= u_0 \quad \text{on } \Omega \times \{0\}.
\end{align*} \]

Here \( c \) is a matrix-valued function which is symmetric and uniformly positive definite on \( \Omega \).
HDG methods for the heat equation.

The approach.

We can obtain \((q, u)\) in \(K \times (0, T)\) in terms of \(\hat{u}\) on \(\partial K \times (0, T)\), \(f\) and \(u_0\) by solving

\[
\begin{align*}
    c q + \nabla u &= 0 \quad \text{in } K \times (0, T), \\
    u_t + \nabla \cdot q &= f \quad \text{in } K \times (0, T), \\
    u &= \hat{u} \quad \text{on } \partial K \times (0, T), \\
    u &= u_0 \quad \text{on } K \times \{0\}.
\end{align*}
\]

The function \(\hat{u}\) can now be determined as the solution on each \(F \times (0, T)\), \(F \in \mathcal{E}_h\), of the equations

\[
\begin{align*}
    [\hat{q}] &= 0 \quad \text{if } F \in \mathcal{E}_h^o, \\
    \hat{u} &= u_D \quad \text{if } F \in \mathcal{E}_h^\partial,
\end{align*}
\]

where \(\hat{q}\) is the trace of \(q = q(\hat{u}, f, u_0)\) on \(\partial K\).
HDG methods for the heat equation.
The semidiscrete method.

At any time, the approximate solution \((q_h, u_h, \hat{u}_h)\) is an element of the space \(V_h \times W_h \times M_h\). It satisfies the equations

\[
\begin{align*}
(c q_h, v)_{\Omega_h} - (u_h, \nabla \cdot v)_{\Omega_h} + \langle \hat{u}_h, v \cdot n \rangle_{\partial \Omega_h} &= 0, \\
((u_h)_t, \nabla w)_{\Omega_h} - (q_h, \nabla w)_{\Omega_h} + \langle \hat{q}_h \cdot n, w \rangle_{\partial \Omega_h} &= (f, w)_{\Omega_h}, \\
\langle \mu, \hat{q}_h \cdot n \rangle_{\partial \Omega_h \setminus \partial \Omega} &= 0, \\
\langle \mu, \hat{u}_h \rangle_{\partial \Omega} &= \langle \mu, u_D \rangle_{\partial \Omega},
\end{align*}
\]

for all \((v, w, \mu) \in V_h \times W_h \times M_h\), where

\[
\hat{q}_h \cdot n = q_h \cdot n + \tau (u_h - \hat{u}_h) \quad \text{on} \ \partial \Omega_h.
\]

The HDG method retains all the convergence and superconvergence, uniformly in time, of the HDG method for the steady-state case provided the initial condition is properly defined.
To approximate the time derivative at time \( t^n := n\Delta t \), we could use the BDF approximation

\[
(u_h)_t^n \approx \left( \sum_{j=0}^{\ell} \gamma_j u_h^{n-j} \right) / \Delta t,
\]

and set

\[
\tilde{f}^n = f^n - \left( \sum_{j=1}^{\ell} \gamma_j u_h^{n-j} \right) / \Delta t,
\]
Then, at any time $t^n = n \Delta t$, the approximate solution $(q_h, u_h, \hat{u}_h)$ is an element of the space $V_h \times W_h \times M_h$. It satisfies the equations

\[
\begin{align*}
(c q_h, v)_{\Omega_h} - (u_h, \nabla \cdot v)_{\Omega_h} + \langle \hat{u}_h, v \cdot n \rangle_{\partial \Omega_h} &= 0, \\
\gamma_0 \frac{\gamma_0}{\Delta t} (u_h, \nabla w)_{\Omega_h} - (q_h, \nabla w)_{\Omega_h} + \langle \hat{q}_h \cdot n, w \rangle_{\partial \Omega_h} &= (\tilde{f}, w)_{\Omega_h}, \\
\langle \mu, \hat{q}_h \cdot n \rangle_{\partial \Omega_h \setminus \partial \Omega} &= 0, \\
\langle \mu, \hat{u}_h \rangle_{\partial \Omega} &= \langle \mu, u_D \rangle_{\partial \Omega},
\end{align*}
\]

for all $(v, w, \mu) \in V_h \times W_h \times M_h$, where

\[
\hat{q}_h \cdot n = q_h \cdot n + \tau(u_h - \hat{u}_h) \quad \text{on } \partial \Omega_h.
\]
Consider the model problem:

\[ u_{tt} + \nabla \cdot (c \nabla u) = f \quad \text{in } \Omega \times (0, T), \]
\[ \hat{u} = (u_D) \quad \text{on } \partial \Omega \times (0, T), \]
\[ u = u_0 \quad \text{on } \Omega \times \{0\}, \]
\[ u_t = u_1 \quad \text{on } \Omega \times \{0\}. \]

Here \( c \) is a matrix-valued function which is symmetric and uniformly positive definite on \( \Omega \).
HDG methods for the wave equation.

The model problem.

We rewrite it in terms of \((q, \nu) := (-c\nabla u, u_t)\) as follows:

\[
\begin{align*}
  c \, q_t + \nabla \nu &= 0 & \text{in } \Omega \times (0, \, T), \\
  \nu_t + \nabla \cdot q &= f & \text{in } \Omega \times (0, \, T), \\
  \nu &= (u_D)_t & \text{on } \partial\Omega \times (0, \, T), \\
  c \, q &= -\nabla u_0 & \text{on } \Omega \times \{0\}, \\
  \nu &= u_1 & \text{on } \Omega \times \{0\}.
\end{align*}
\]

Here \(c\) is a matrix-valued function which is symmetric and uniformly positive definite on \(\Omega\).
HDG methods for the wave equation.

The approach.

We can obtain \((q, v)\) in \(K \times (0, T)\) in terms of \(\hat{v}\) on \(\partial K \times (0, T)\), \(f\), \(u_0\) and \(u_1\) by solving

\[
c q_t + \nabla u = 0 \quad \text{in} \ K \times (0, T),
\]
\[
v_t + \nabla \cdot q = f \quad \text{in} \ K \times (0, T),
\]
\[
c q = -\nabla u_0 \quad \text{on} \ \Omega \times \{0\},
\]
\[
v = u_1 \quad \text{on} \ \Omega \times \{0\}.
\]

The function \(\hat{v}\) can now be determined as the solution on each \(F \times (0, T)\), \(F \in \mathcal{E}_h\), of the equations

\[
[\hat{q}] = 0 \quad \text{if} \ F \in \mathcal{E}_h^o,
\]
\[
\hat{v} = (u_D)_t \quad \text{if} \ F \in \mathcal{E}_h^\partial,
\]

where \(\hat{q}\) is the trace of \(q = q(\hat{u}, f, u_0, u_1)\) on \(\partial K\).
HDG methods for the wave equation.
The semidiscrete method.

At any time, the approximate solution \((q_h, v_h, \hat{v}_h)\) is an element of the space \(V_h \times W_h \times M_h\). It satisfies the equations

\[
\begin{align*}
(c(q_h)_t, r)_{\Omega_h} - (v_h, \nabla \cdot r)_{\Omega_h} + \langle \hat{v}_h, r \cdot n \rangle_{\partial \Omega_h} &= 0, \\
((v_h)_t, \nabla w)_{\Omega_h} - (q_h, \nabla w)_{\Omega_h} + \langle \hat{q}_h \cdot n, w \rangle_{\partial \Omega_h} &= (f, w)_{\Omega_h}, \\
\langle \mu, \hat{q}_h \cdot n \rangle_{\partial \Omega_h \setminus \partial \Omega} &= 0, \\
\langle \mu, \hat{v}_h \rangle_{\partial \Omega} &= \langle \mu, (u_D)_t \rangle_{\partial \Omega},
\end{align*}
\]

for all \((r, w, \mu) \in V_h \times W_h \times M_h\), where

\[
\hat{q}_h \cdot n = q_h \cdot n + \tau(v_h - \hat{v}_h) \quad \text{on } \partial \Omega_h.
\]
The semidiscrete method.

For simplexes, $\mathbf{V}(K) := \mathcal{P}_k(K)$ and $\mathcal{W}(K) := \mathcal{P}_k(K)$:

- The HDG method converges in $q_h$ and $v_h$ with the optimal order of $k + 1$, for $k \geq 0$, in the $L^\infty(0, T; L^2(\Omega))$-norm.
- The variable $\int_0^t v_h$ superconverges with order $k + 2$, for $k \geq 1$, in the $L^\infty(0, T; L^2(\Omega))$-norm provided the initial conditions are suitably defined.
- In this case, the postprocessed solution $u^*_h$ superconverges with order $k + 2$, for $k \geq 1$, in the $L^\infty(0, T; L^2(\Omega))$-norm.

Recall that, on each element $K$, $u^*_h$ lies in the space $\mathcal{P}_{k+1}(K)$ and is defined by

$$
(\nabla u^*_h, \nabla w)_K = - (c q_h, \nabla w)_K \quad \text{for all } w \in \mathcal{P}_{k+1}(K),
$$

$$
(u^*_h, 1)_K = (u_h, 1)_K = \left( \int_0^t v_h + u_h(0), 1 \right)_K.
$$
Consider the model problem:

\[ c \mathbf{q} + \nabla u = 0 \quad \text{in } \Omega \times (0, T), \]
\[ \nabla \cdot (\mathbf{q} + \mathbf{v} u) = f \quad \text{in } \Omega \times (0, T), \]
\[ \hat{u} = u_D \quad \text{on } \partial \Omega \times (0, T). \]

Here \( c \) is a matrix-valued function which is symmetric and uniformly positive definite on \( \Omega \).
HDG methods for convection-diffusion equations.

The approach.

We can obtain \((q, u)\) in \(K \times (0, T)\) in terms of \(\hat{u}\) on \(\partial K \times (0, T)\), \(f\) and \(u_0\) by solving

\[
\begin{align*}
\mathbf{c} \cdot \mathbf{q} + \nabla u &= 0 \quad \text{in } K \times (0, T), \\
\nabla \cdot (\mathbf{q} + \mathbf{v} \cdot u) &= f \quad \text{in } K \times (0, T), \\
u &= \hat{u} \quad \text{on } \partial K \times (0, T).
\end{align*}
\]

The function \(\hat{u}\) can now be determined as the solution on each \(F \times (0, T), F \in \mathcal{E}_h\), of the equations

\[
\begin{align*}
[\hat{q} + \mathbf{v} \cdot \hat{u}] &= 0 \quad \text{if } F \in \mathcal{E}_h^o, \\
\hat{u} &= u_D \quad \text{if } F \in \mathcal{E}_h^\partial,
\end{align*}
\]

where \(\hat{q}\) is the trace of \(q = q(\hat{u}, f, u_0)\) on \(\partial K\).
HDG methods for convection-diffusion

Definition of the method.

The HDG method defines the approximation \((q_h, u_h, \hat{u}_h)\) in \(V_h \times W_h \times M_h\) by requiring that

\[
(c \, q_h, r)_{\Omega_h} - (u_h, \nabla \cdot r)_{\Omega_h} + \langle \hat{u}_h, r \cdot n \rangle_{\partial \Omega_h} = 0,
\]

\[
-(q_h + u_h v, \nabla w)_{\Omega_h} + \langle (\hat{q}_h + \hat{u}_h v) \cdot n, w \rangle_{\partial \Omega_h} = (f, w)_{\Omega_h},
\]

\[
\langle \mu, \hat{u}_h \rangle_{\partial \Omega} = \langle \mu, g \rangle_{\partial \Omega},
\]

\[
\langle \mu, (\hat{q}_h + \hat{u}_h v) \cdot n \rangle_{\partial \Omega_h \setminus \partial \Omega} = 0,
\]

hold for all \((r, w, \mu) \in V_h \times W_h \times M_h\), where

\[
\hat{q}_h + \hat{u}_h v = q_h + \hat{u}_h v + \tau(u_h - \hat{u}_h)n \quad \text{on} \quad \partial \Omega_h.
\]
Definition of the method.

**Theorem**

*The method is well defined if*

**A1** There is a constant $\gamma_0 > 0$: $\min(\tau - \frac{1}{2} \mathbf{v} \cdot \mathbf{n})|_{\partial K} \geq \gamma_0 \forall K \in \mathcal{T}_h$.

**A2** On any face $F \in \mathcal{E}_h$, $\tau$ is a constant.

The following practical choices of stabilization functions $\tau$ do satisfy these two conditions:

$$\tau^+ = \tau^- = |\mathbf{v} \cdot \mathbf{n}| + \frac{\kappa}{\ell},$$

$$(\tau^+, \tau^-) = \begin{cases} (|\mathbf{v} \cdot \mathbf{n}| + \frac{\kappa}{\ell}, 0) & \text{when } \mathbf{v} \cdot \mathbf{n}^- \leq 0, \\ (0, |\mathbf{v} \cdot \mathbf{n}| + \frac{\kappa}{\ell}) & \text{when } \mathbf{v} \cdot \mathbf{n}^- > 0. \end{cases}$$

Here $\kappa$ is a scalar proportional to some norm of the diffusivity matrix $c^{-1}$ and $\ell$ denotes a representative length scale.
HDG methods for convection-diffusion

The numerical traces.

For the first choice of $\tau$, we have

$$\hat{u}_h = \{u_h\} + \frac{1}{2\tau} \left[ q_h \cdot n \right],$$

$$\hat{u}_h v + \hat{q}_h = \{u_h\} v + \{q_h\} + \frac{1}{2\tau} \left[ q_h \cdot n \right] v + \frac{\tau}{2} \left[ u_h n \right],$$

whereas for the second choice for $\tau$,

$$\begin{cases} 
\hat{u}_h = u_h^+ + \frac{1}{\tau^+} \left[ q_h \cdot n \right], \\
\hat{u}_h v + \hat{q}_h = u_h^+ v + q_h^- + \frac{1}{\tau^+} \left[ q_h \cdot n \right] v 
\end{cases} \quad \text{if } v \cdot n^- \leq 0,$$

and

$$\begin{cases} 
\hat{u}_h = u_h^- + \frac{1}{\tau^-} \left[ q_h \cdot n \right], \\
\hat{u}_h v + \hat{q}_h = u_h^- v + q_h^+ + \frac{1}{\tau^-} \left[ q_h \cdot n \right] v, 
\end{cases} \quad \text{if } v \cdot n^- > 0.$$

Numerical examples.

Unstructured and anisotropic meshes.

Numerical examples.

HDG approximation with quadratic polynomials on the unstructured triangulation.

Numerical examples.

HDG approximation with quadratic polynomials on the unstructured triangulation.
Linear elasticity. (S.-C.Soon, B.C. and H.Stolarski, JNME, 2009.)

The model problem.

Consider the following problem:

\[
\begin{align*}
\sigma_{ij,j} + b_i &= 0 \quad \text{in } \Omega, \\
\epsilon_{ij} - \frac{1}{2} (u_{i,j} + u_{j,i}) &= 0 \quad \text{in } \Omega, \\
\sigma_{ij} - D_{ijkl} \epsilon_{kl} &= 0 \quad \text{in } \Omega, \\
\hat{u}_i &= u_i \quad \text{on } \partial\Omega_D, \\
\hat{\sigma}_{ij} n_j &= t_i \quad \text{on } \partial\Omega_N.
\end{align*}
\]
We can obtain \((\sigma, u)\) in \(K\) in terms of \(\hat{u}\) by solving

\[
\begin{align*}
\sigma_{ij,j} + b_i &= 0 \quad \text{in } K, \\
\epsilon_{ij} - \frac{1}{2} (u_{i,j} + u_{j,i}) &= 0 \quad \text{in } K, \\
\sigma_{ij} - D_{ijkl} \epsilon_{kl} &= 0 \quad \text{in } K, \\
\hat{u}_i &= \hat{u}_i \quad \text{on } \partial K.
\end{align*}
\]

The function \(\hat{u}\) can now be determined as the solution of the transmission condition

\[
\begin{align*}
[\sigma_{ij} n_j] &= 0 \quad \text{on } \mathcal{E}^\circ_h, \\
\hat{u}_i &= u_i \quad \text{on } \partial \Omega_D, \\
\hat{\sigma}_{ij} n_j &= t_i \quad \text{on } \partial \Omega_N.
\end{align*}
\]
Linear elasticity.
An HDG method

The approximation \((u^h, \sigma^h, \epsilon^h, \hat{u}^h)\) is taken in the finite dimensional space \(V^h \times W^h \times Z^h \times M^h\) where

\[
V^h = \{ v \in L^2(\Omega) : v_i \mid_K \in \mathcal{P}_k(K) \quad \forall K \in \Omega_h, \quad i = 1, 2, 3 \},
\]

\[
W^h = \{ w \in L^2(\Omega) : w_{ij} \mid_K \in \mathcal{P}_k(K) \quad \forall K \in \Omega_h, \quad i, j = 1, 2, 3 \},
\]

\[
Z^h = \{ z \in L^2(\Omega) : z_{ij} \mid_K \in \mathcal{P}_k(K) \quad \forall K \in \Omega_h, \quad i, j = 1, 2, 3 \},
\]

\[
M^h = \{ \mu \in L^2(\mathcal{E}_h) : \mu_i \mid_F \in \mathcal{P}_k(F) \quad \forall F \in \mathcal{E}_h, \quad i = 1, 2, 3 \}.
\]
Linear elasticity.
An HDG method.

On the element $K$, $(\mathbf{u}^h, \sigma^h, \epsilon^h)$ is obtained in terms of $\hat{\mathbf{u}}^h$ by solving

$$
(v_{ij}, \sigma_{ij}^h)_K - \langle v_i, \hat{\sigma}_{ij}^h n_j \rangle_{\partial K} - (v_i, b_i)_K = 0,
$$

$$
(w_{ij}, \epsilon_{ij}^h)_K - \frac{1}{2} \langle w_{ij}, (\hat{u}^h_i n_j + \hat{u}^h_j n_i) \rangle_{\partial K} + \frac{1}{2} (w_{ij}, u_i^h)_K + \frac{1}{2} (w_{ij}, u_j^h)_K = 0,
$$

$$
(z_{ij}, \sigma_{ij}^h)_K - (z_{ij}, D_{ijkl} \epsilon_{kl}^h)_K = 0,
$$

for all $(v, w, z, \mu) \in \mathcal{P}_k(K) \times \mathcal{P}_k(K) \times \mathcal{P}_k(K) \times \mathcal{P}_k(K)$, where

$$
\hat{\sigma}_{ij}^h = \sigma_{ij}^h - \tau_{ijkl} (u_k^h - \hat{u}_k^h) n_l \quad \text{on} \quad \partial \Omega_h.
$$

The function $\hat{\mathbf{u}}^h$ is now determined as the element of $\mathbf{M}_h$ satisfying

$$
\langle \mu_i, \hat{\sigma}_{ij}^h n_j \rangle_{\partial \Omega_h \setminus \partial \Omega_D} = \langle \mu_i, t_i \rangle_{\partial \Omega_N},
$$

$$
\langle \mu_i, \hat{u}_j^h \rangle_{\partial \Omega_D} = \langle \mu_i, u_i \rangle_{\partial \Omega_D}.
$$

for all $\mu \in \mathbf{M}_h$. 

Bernardo Cockburn (UMN, KFUPM) 
HDG Methods. Part I 
Dhahran, 2012 24 / 50
Linear elasticity.

An HDG method

In compact form, the methods can be written as follows:

\[
\begin{align*}
(v_{i,j}, \sigma_{ij}^h)_{\Omega_h} - \langle v_i, \hat{\sigma}_{ij}^h n_j \rangle_{\partial\Omega_h} - (v_i, b_i)_{\Omega_h} &= 0, \\
(w_{ij}, \epsilon_{ij}^h)_{\Omega_h} - \frac{1}{2} \langle w_{ij}, (\hat{u}_i^h n_j + \hat{u}_j^h n_i) \rangle_{\partial\Omega_h} + \frac{1}{2} \left( w_{ij}, u_i^h \right)_{\Omega_h} + \frac{1}{2} \left( w_{ij}, u_j^h \right)_{\Omega_h} &= 0, \\
(z_{ij}, \sigma_{ij}^h)_{\Omega_h} - \left( z_{ij}, D_{ijkl} \epsilon_{kl}^h \right)_{\Omega_h} &= 0, \\
\langle \mu_i, \hat{\sigma}_{ij}^h n_j \rangle_{\partial\Omega_h \setminus \partial\Omega_D} &= \langle \mu_i, t_i \rangle_{\partial\Omega_N}, \\
\langle \mu_i, \hat{u}_i^h \rangle_{\partial\Omega_D} &= \langle \mu_i, u_i \rangle_{\partial\Omega_D},
\end{align*}
\]

for all \((v, w, z, \mu) \in V^h \times W^h \times Z^h \times M^h\), where

\[
\hat{\sigma}_{ij}^h = \sigma_{ij}^h - \tau_{ijkl} (u_k^h - \hat{u}_k^h) n_l \quad \text{on } \partial\Omega_h.
\]
Linear elasticity.

An HDG method

In compact form:

\[
\begin{align*}
(v_{i,j}, \sigma_{ij}^h)_{\Omega_h} - \langle v_i, \hat{\sigma}_{ij}^h n_j \rangle_{\partial \Omega_h} - (v_i, b_i)_{\Omega_h} &= 0, \\
(w_{ij}, \epsilon_{ij}^h)_{\Omega_h} - \frac{1}{2} \langle w_{ij}, (\hat{u}_i^h n_j + \hat{u}_j^h n_i) \rangle_{\partial \Omega_h} + \frac{1}{2} (w_{ij,j}, u_i^h)_{\Omega_h} + \frac{1}{2} (w_{ij,i}, u_j^h)_{\Omega_h} &= 0, \\
(z_{ij}, \sigma_{ij}^h)_{\Omega_h} - (z_{ij}, D_{ijkl} \epsilon_{kl}^h)_{\Omega_h} &= 0, \\
\langle \mu_i, \hat{\sigma}_{ij}^h n_j \rangle_{\partial \Omega_h \setminus \partial \Omega_D} &= \langle \mu_i, t_i \rangle_{\partial \Omega_N}, \\
\langle \mu_i, \hat{u}_i^h \rangle_{\partial \Omega_D} &= \langle \mu_i, u_i \rangle_{\partial \Omega_D},
\end{align*}
\]

for all \((v, w, z, \mu) \in V^h \times W^h \times Z^h \times M^h\), where

\[
\hat{\sigma}_{ij}^h = \sigma_{ij}^h - \tau_{ijkl} (u^h_k - \hat{u}_k^h) n_l \\
\text{on } \partial \Omega_h.
\]
The approximate solution

\[
(u^h, \sigma^h, \epsilon^h) = (U(\hat{u}^h), S(\hat{u}^h), E(\hat{u}^h)) + (U(u), S(u), E(u)),
\]

is well defined if we take \(\tau_{ijkl} n_j n_l\) positive definite on \(\partial\Omega_h\). Moreover, the function \(\lambda^h := \hat{u}^h - u\), is the only element of \(M^h\) satisfying

\[
a^h (\mu, \lambda^h) = b^h (\mu) \quad \forall \mu \in M^h(0),
\]

where

\[
a^h (\zeta, \eta) = \left( D_{ijkl} E^{(\zeta)}_{ij}, E^{(\eta)}_{kl} \right)_{\Omega_h} + \left\langle \left( U^{(\eta)}_i - \eta_i \right), \tau_{ijkl} n_j n_l \left( U^{(\zeta)}_k - \zeta_k \right) \right\rangle_{\partial\Omega_h},
\]

\[
b^h (\zeta) = \left\langle \zeta_i, t_i \right\rangle_{\partial\Omega_N} - \left\langle \hat{S}^{(\zeta)}_{ij} n_j, u_i \right\rangle_{\partial\Omega_D} + \left( U^{(\zeta)}_i, b_i \right)_{\Omega_h},
\]

for all \(\zeta, \eta \in L^2(\mathcal{E}^h)\).
Linear elasticity.
Numerical experiments.

- For $k \geq 0$ all unknowns converge with order $k + 1$.
- For $k \geq 2$ the local average of the displacement superconverges with order $k + 2$. A local postprocessing can be devised that provides another approximate displacement converging with order $k + 2$.
- Analysis: Still open!
Consider the model problem:

\[-\nu \Delta u + \nabla p = f \quad \text{in } \Omega,\]
\[\nabla \cdot u = 0 \quad \text{on } \Omega,\]
\[\hat{u} = u_D \quad \text{on } \partial \Omega,\]

where \(\langle u_D \cdot n, 1 \rangle_{\partial \Omega} = 0\) and \((p, 1)_{\Omega} = 0\).
HDG methods for the Stokes flow.
Using the velocity gradient.

We begin by rewriting it as follows:

\[ \begin{align*}
L - \nabla u &= 0 \quad \text{in } \Omega, \\
-\nu \nabla \cdot L + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{on } \Omega, \\
\hat{u} &= u_D \quad \text{on } \partial\Omega,
\end{align*} \]

where \( \langle u_D \cdot n, 1 \rangle_{\partial\Omega} = 0 \) and \( (p, 1)_\Omega = 0 \).
HDG methods for the Stokes flow
Using the velocity gradient.

We can express \((L, u, p)\) in \(K\) in terms of \(\hat{u}\) on \(\partial K\) and \(\bar{p} := (p, 1)_{K}/|K|\) by solving

\[
\begin{align*}
L - \nabla u &= 0, & -\nu \nabla \cdot L + \nabla p &= f \quad \text{in } K, \\
\nabla \cdot u &= \frac{1}{|K|} \langle \hat{u} \cdot n, 1 \rangle_{\partial K} \quad \text{in } K, \\
u \nu \nabla \cdot L + \nabla p &= f \quad \text{in } K, \\
\hat{u} &= \hat{u} \quad \text{on } \partial K.
\end{align*}
\]

The functions \(\hat{u}\) and \(\bar{p}\) are the solution of

\[
\begin{align*}
\left[ -\nu \hat{L} n + \hat{p} n \right] &= 0 \quad \text{for all } F \in \mathcal{E}_h^o, \\
\langle \hat{u} \cdot n, 1 \rangle_{\partial K} &= 0 \quad \text{for all } K \in \Omega_h, \\
\hat{u} &= u_D \quad \text{on } \partial \Omega, \\
(\bar{p}, 1)_{\Omega} &= 0.
\end{align*}
\]
On the element $K \in \Omega_h$, we define $(L_h, u_h, p_h)$ in terms of $(\hat{u}_h, \bar{p}_h, f)$ as the element of $G(K) \times V(K) \times Q(K)$ solving

\[
(L_h, G)_K + (u_h, \nabla \cdot G)_K - \langle \hat{u}_h, Gn \rangle_{\partial K} = 0, \\
\nu (L_h, \nabla v)_K - (p_h, \nabla \cdot v)_K - \langle \nu \hat{L}_h n - \bar{p}_h n, v \rangle_{\partial K} = \langle f, v \rangle_K, \\
- (u_h, \nabla q)_{\Omega_h} + \langle \hat{u}_h \cdot n, q - \bar{q} \rangle_{\partial K} = 0,
\]

for all $(G, v, q) \in G(K) \times V(K) \times Q(K)$, where

\[
-\nu \hat{L}_h n + \bar{p}_h n = -\nu L_h n + p_h n + \nu \tau (u_h - \hat{u}_h) \quad \text{on } \partial K,
\]

and $(p_h, 1)_K / |K| = \bar{p}_h$. 

The HDG methods for the Stokes flow

The weak formulation for \((\hat{\mathbf{u}}_h, \overline{p}_h, f)\).

We take \(\hat{\mathbf{u}}_h|_F\) in \(\mathbf{M}(F)\) and \(\overline{p}_h|_K\) in \(\mathcal{P}_0(K)\) and determine them by requiring

\[
\begin{align*}
\langle \left[ -\nu \hat{L}_h \mathbf{n} + \hat{p}_h \mathbf{n} \right], \mu \rangle_F &= 0 & \forall \mu \in \mathbf{M}(F) \quad \forall F \in \mathcal{E}^o_h, \\
\langle \hat{\mathbf{u}}_h \cdot \mathbf{n}, 1 \rangle_{\partial K} &= 0 & \forall K \in \Omega_h, \\
\langle \hat{\mathbf{u}}_h, \mu \rangle_F &= \langle \mathbf{u}_D, \mu \rangle_F & \forall \mu \in \mathbf{M}(F) \quad \forall F \in \mathcal{E}^\partial_h, \\
\left( \overline{p}_h, 1 \right)_\Omega &= 0.
\end{align*}
\]
The HDG methods for the Stokes flow
Existence and Uniqueness.

Theorem

The HDG methods are well defined if

1. $\tau > 0$ on $\partial \Omega_h$,
2. $\nabla \mathbf{V}(K) \in G(K) \quad \forall K \in \Omega_h$,
3. $\nabla Q(K) \in V(K) \quad \forall K \in \Omega_h$. 
We denote by \((L, U, P)\) the linear mapping that associates \((\hat{u}_h, \bar{p}_h, f)\) to \((L_h, u_h, p_h)\), and set

\[
\begin{align*}
(L \hat{u}_h, U \hat{u}_h, P \hat{u}_h) & := (L, U, P)(\hat{u}_h, 0, 0), \\
(L \bar{p}_h, U \bar{p}_h, P \bar{p}_h) & := (L, U, P)(0, \bar{p}_h, 0), \\
(L f, U f, P f) & := (L, U, P)(0, 0, f).
\end{align*}
\]

Then we have that

\[
(L_h, u_h, p_h) = (L \hat{u}_h, U \hat{u}_h, P \hat{u}_h) + (L \bar{p}_h, U \bar{p}_h, P \bar{p}_h) + (L f, U f, P f).
\]
The HDG methods for the Stokes flow
Implementation. Characterization of $\hat{u}_h$ and $\bar{p}_h$

The function $(\hat{u}_h, \bar{p}_h)$ is the only element in $M_h \times \bar{P}_h$ such that

$$a_h(\hat{u}_h, \mu) + b_h(\bar{p}_h, \mu) = \ell_h(\mu), \quad \forall \mu \in M_h : \mu|_{\partial \Omega} = 0,$$

$$b_h(q, \hat{u}_h) = 0, \quad \forall q \in \bar{P}_h,$$

$$\hat{u}_h = u_D,$$

$$\bar{p}_h(1)_\Omega = 0.$$

where $M_h := \{\mu \in L^2(\mathcal{E}_h) : \mu|_F \in M(F) \ \forall \ F \in \mathcal{E}_h^o\}$.

- The bilinear form $a_h(\cdot, \cdot)$ is symmetric and positive definite on $M_{h,0} \times M_{h,0}$. 

Bernardo Cockburn (UMN, KFUPM) HDG Methods. Part I Dhahran, 2012 36 / 50
The HDG methods for the Stokes flow.

Compact form of the HDG methods.

\((L_h, u_h, p_h, \hat{u}_h)\) is the element of \(G_h \times V_h \times Q_h \times M_h\) solving

\[
(L_h, G)_{\Omega_h} + (u_h, \nabla \cdot G)_{\Omega_h} - \langle \hat{u}_h, G \mathbf{n} \rangle_{\partial \Omega_h} = 0,
\]

\[
(\nu L_h, \nabla v)_{\Omega_h} - (p_h, \nabla \cdot v)_{\Omega_h} - \langle \nu \hat{L}_h \mathbf{n} - \hat{p}_h \mathbf{n}, v \rangle_{\partial \Omega_h} = (f, v)_{\Omega_h},
\]

\[
-(u_h, \nabla q)_{\Omega_h} + \langle \hat{u}_h \cdot n, q \rangle_{\partial \Omega_h} = 0,
\]

\[
\langle -\nu \hat{L}_h \mathbf{n} + \hat{u}_h \hat{u}_h \cdot \mathbf{n} + \hat{p}_h \mathbf{n}, \mu \rangle_{\partial \Omega_h \setminus \partial \Omega} = 0.
\]

\[
\langle \hat{u}_h, \mu \rangle_{\partial \Omega} = \langle u_D, \mu \rangle_{\partial \Omega}
\]

\[
(p_h, 1)_{\Omega} = 0,
\]

for all \((G, v, q, \mu) \in G_h \times V_h \times Q_h \times M_h\), where

\[
-\nu \hat{L}_h \mathbf{n} + \hat{p}_h \mathbf{n} = -\nu L_h \mathbf{n} + p_h \mathbf{n} + \nu \tau (u_h - \hat{u}_h) \quad \text{on} \ \partial \Omega_h.
\]
The energy identity for the exact solution is

\[(L, L)_\Omega = (f, u)_\Omega + \langle -\nu L n + p n, u_D \rangle_{\partial\Omega},\]

and for the approximate solution we have,

\[(L_h, L_h)_\Omega + \Theta_\tau(u_h - \hat{u}_h) = (f, u_h)_\Omega + \langle (-\nu \hat{L}_h + \hat{p}_h I)n, u_D \rangle_{\partial\Omega},\]

where \(\Theta_\tau(u_h - \hat{u}_h) := \sum_{K \in \Omega_h} \langle \tau(u_h - \hat{u}_h), u_h - \hat{u}_h \rangle_{\partial K}.\) We see that the jumps \(u_h - \hat{u}_h\) stabilize the method if we require the function \(\tau\) to be positive on \(\partial\Omega_h\).
The HDG methods for the Stokes flow

The stabilization mechanism. The jumps of the velocity control the residuals.

The Galerkin formulation on the element $K$ reads

$$
(R^u_K, G)_K = \langle R^u_{\partial K}, G \rangle_{\partial K}
$$

$$
(R^{L,p}_K, v)_K = \langle R^{L,p}_{\partial K}, v \rangle_{\partial K},
$$

$$
(R^{\nabla \cdot u}_K, q)_K = \langle tr R^u_{\partial K}, q \rangle_{\partial K},
$$

for all $(G, v, q) \in G(K) \times V(K) \times P(K)$ where

$$
R^u_K := L_h - \nabla u_h,
$$

$$
R^{L,p}_K := \nabla \cdot (-\nu L_h + p_h I) - f,
$$

$$
R^{\nabla \cdot u}_K := \nabla \cdot u_h,
$$

$$
R^u_{\partial K} := (\tilde{u}_h - u_h) \otimes n,
$$

$$
R^{L,p}_{\partial K} := (-\nu L_h n + p_h n) - (-\nu \tilde{L}_h n + \tilde{p}_h n) = -\nu \tau (u_h - \tilde{u}_h)$$

Construction of superconvergent HDG methods.

- Let $\mathbf{V}^D(K)$, $W^D(K)$ and $M^D(F)$ be the local spaces of a superconvergent HDG method for diffusion.

- Set $G_i(K) := V^D(K)$, $V_i(K) := W^D(K)$ and $M_i(F) := M^D(F)$.

- Take a local space $Q(K)$ such that

$$\nabla \cdot \mathbf{V}(K) \subset Q(K), \quad Q(K) \cap \subset G(K).$$
The HDG methods for the Stokes flow
Convergence properties.

**Theorem**

We have

\[ \| E^L \|_\Omega \leq C \| \Pi L - L \|_\Omega, \]
\[ \| \varepsilon^p \|_\Omega \leq C \sqrt{C_T} \nu \| \Pi L - L \|_\Omega, \]

where \( C_T := \max_{K \in \Omega_h} \{1, \tau_K h_K \} \). Moreover,

\[ \| \varepsilon_u \|_\Omega \leq C C_T h^{\min\{k, 1\}} \| \Pi L - L \|_\Omega, \]

provided a standard elliptic regularity result holds.

Note that, by an energy argument, we get

\[ (E^L, E^L)_\Omega + \Theta_T (\varepsilon_u - \varepsilon_\widehat{u}) = (\Pi L - L, E^L)_\Omega. \]
A new approximate velocity $\mathbf{u}_h^\star$ can be obtained which has the following properties:

- It is computed in an element-by-element fashion.
- $\mathbf{u}_h^\star \in \mathbf{H}(\text{div}, \Omega)$.
- $\nabla \cdot \mathbf{u}_h^\star = 0$ on $\Omega$.
- $\|\mathbf{u}_h^\star - \mathbf{u}\|_\Omega \leq C C_{\tau} h^{\min\{k,1\}} \|\Pi L - L\|_\Omega + C h^{k+2} \|\mathbf{u}\|_{H^{k+2}(\Omega)}$.

The model problem.

Consider the model problem:

\[-\nu \Delta u + \nabla \cdot (u \otimes u) + \nabla p = f \text{ in } \Omega,
\]
\[\nabla \cdot u = 0 \text{ on } \Omega,
\]
\[\hat{u} = u_D \text{ on } \partial \Omega,
\]

where \( \langle u_D \cdot n, 1 \rangle_{\partial \Omega} = 0 \) and \( (p, 1)_\Omega = 0 \).
The incompressible Navier-Stokes equations.

Compact form of the HDG methods.

\((\mathbf{L}_h, \mathbf{u}_h, \mathbf{p}_h, \mathbf{\hat{u}}_h)\) is the element of \(G_h \times \mathbf{V}_h \times Q_h \times M_h\) solving

\[
(\mathbf{L}_h, G)_{\Omega_h} + (\mathbf{u}_h, \nabla \cdot G)_{\Omega_h} - \langle \mathbf{\hat{u}}_h, Gn \rangle_{\partial \Omega_h} = 0,
\]

\[
(\nu \mathbf{L}_h, \nabla \mathbf{v})_{\Omega_h} - (\mathbf{u}_h \otimes \mathbf{u}_h, \nabla \mathbf{v})_{\Omega_h}
\]

\[
- (\mathbf{p}_h, \nabla \cdot \mathbf{v})_{\Omega_h} - \langle \nu \mathbf{L}_h n + \mathbf{\hat{u}}_h \mathbf{\hat{u}}_h \cdot n - \mathbf{\hat{p}}_h n, \mathbf{v} \rangle_{\partial \Omega_h} = (\mathbf{f}, \mathbf{v})_{\Omega_h},
\]

\[
- (\mathbf{u}_h, \nabla q)_{\Omega_h} + \langle \mathbf{\hat{u}}_h \cdot n, q \rangle_{\partial \Omega_h} = 0,
\]

\[
\langle -\nu \mathbf{L}_h n + \mathbf{\hat{u}}_h \mathbf{\hat{u}}_h \cdot n + \mathbf{\hat{p}}_h n, \mu \rangle_{\partial \Omega_h \setminus \partial \Omega} = 0
\]

\[
\langle \mathbf{\hat{u}}_h, \mu \rangle_{\partial \Omega} = \langle \mathbf{u}_D, \mu \rangle_{\partial \Omega}
\]

\[
(\mathbf{p}_h, 1)_{\Omega} = 0,
\]

for all \((G, \mathbf{v}, q, \mu) \in G_h \times \mathbf{V}_h \times Q_h \times M_h\), where

\[
-\nu \mathbf{L}_h n + \mathbf{\hat{p}}_h n = -\nu \mathbf{L}_h n + \mathbf{p}_h n + \nu \tau (\mathbf{u}_h - \mathbf{\hat{u}}_h) \quad \text{on } \partial \Omega_h.
\]
The compressible Navier-Stokes equations.
A numerical example.

Viscous flow over a Kármán-Trefftz airfoil: $M_\infty = 0.1$, $Re = 4000$ and $\alpha = 0$. Mach number distribution (left) and detail of the mesh and Mach number solution near the leading edge region (right) using fourth order polynomial approximations.

(N.-C. Nguyen, J. Peraire and B.C., 2011.)
The Euler equations of gas dynamics.
A numerical example.

Inviscid flow over a Kármán-Trefftz airfoil: $M_{\infty} = 0.1$, $\alpha = 0$. Detail of the mesh employed (left) and Mach number contours of the solution using fourth order polynomial approximations (right).

(N.-C. Nguyen, J. Peraire and B.C., 2011.)
Ongoing work and open problems

- Other stabilization functions? Other choices of local spaces?
- Superconvergence for pyramidal, hexahedral elements?
- A posteriori error estimates: Only in terms of $u_h - \hat{u}_h$ and $\tau$?
- Efficient solvers: Domain decomposition methods?
- Stokes flow: Superconvergence with other formulations?
- Solid mechanics: Optimal convergence for all variables?
- Linear transport: Which unknowns superconverge?
- HDG methods for KdV equations: Superconvergence?
- Nonlinear hyperbolic conservation laws: How to deal with shocks?
The HDG methods.
References after 2005 (I am aware of!).

- **Time-dependent diffusion:**

- **The wave equation:**
  - (B.C. and V.Queneville-Bélair, Math. Comp., 2nd. revision.) Analysis of the semidiscrete case.
  - (X. Feng and Y.Xing, Math. Comp., to appear.) Helmholtz equation for arbitrary frequencies.

- **Convection -diffusion:**
  - (B.C., B.Dong, J.Guzmán, R.Sacco and M.Restelli, SISC, 2009.) Devising an HDG method for the linear, steady-state case.
  - (S. Rheberghen and B.C., "80 years of the CFL Condition", to appear.) Devising space-time HDG methods.
  - (I.Oikawa, submitted.) DG methods of hybrid type.
The HDG methods.
References after 2005 (I am aware of!).

- **Linear and nonlinear elasticity:**
  - (B.C. and K. Shi, IMA, in revision.) Devising and analysis of superconvergent methods for linear elasticity.

- **Stokes flow:**
  - (B.C. and J. Gopalakrishnan, SINUM, 2009.) Devising HDG methods with vorticity formulation.
  - (N.C. Nguyen, J. Peraire and B.C., JCP+CMAME, 2010.) Devising HDG methods with velocity gradient formulation + comparison with other formulations.
The HDG methods.
References after 2005 (I am aware of!).

- **Incompressible Navier-Stokes:**
  - (S. Rhebergen and B.C., JCP, to appear.) Devising space-time HDG methods.

- **Timoshenko beams, biharmonic:**

- **Compressible fluid flow:**